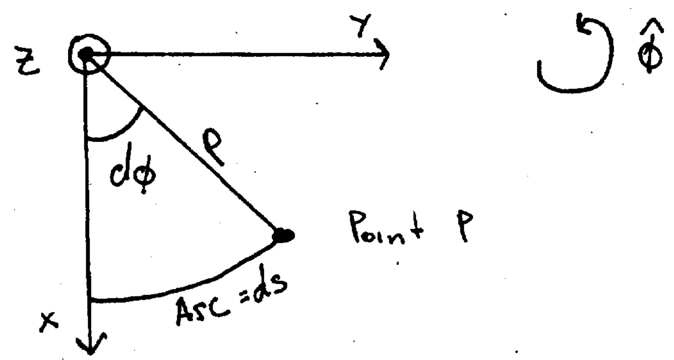
Solutions PS1

1\. a) The first step is to solve for $\vec{v}$. This is most easily done in cylindrical coords.

The vector $\vec{\omega} = \omega \hat{z}$ tells us that $\vec{v}$ is rotating about the z-axis in the $\hat{\phi}$ direction.

Recall from mechanics that $\vec{\omega} = \vec{\rho} \times \vec{v}$.



To calculate the magnitude $\|\vec{v}\|$ we start with $v = \frac{\Delta \text{distance}}{\Delta \text{time}} = \frac{ds}{dt}$. We know that the Arc $ds = \rho d\phi$ so that $v = \rho \frac{d\phi}{dt}$. Since $\omega = \frac{d\phi}{dt}$

$\vec{v} = \rho \omega \hat{\phi}$.

Looking in the cover of Griffiths we see that only the $\hat{z}$ component survives for $\nabla \times \vec{v}$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v_\phi)\hat{z} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\omega \rho^2)\hat{z} = 2\omega\hat{z}$$

1. cont.

b) now $\vec{v} = \rho\omega\hat{\phi} = \frac{a}{\rho}\hat{\phi}$, so again only the $\hat{z}$ component has a chance to survive

$\nabla \times \vec{v} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\phi)\hat{z}$. However, the ρ's cancel

so that $\nabla \times \vec{v} = \frac{1}{\rho}\cancel{\frac{\partial}{\partial\rho} a}^{\,0}\,\hat{z} = 0$

2. a) $\nabla \times \vec{A} = \frac{b}{\rho}\hat{z}$ $\qquad$ $\vec{A} = a\hat{\rho} + b\hat{\phi} + c\hat{z}$

$\vec{\nabla} \cdot \vec{A} = \frac{a}{\rho}$ $\qquad$ Just use front cover of Griffiths

Why is this unique? Well this vector $\vec{A}$ <u>seems</u> like a constant vector field, i.e. if we were to compute $\nabla \times \vec{A}$ and $\nabla \cdot \vec{A}$ for $\vec{A} = a\hat{x} + b\hat{y} + c\hat{z}$ they would both vanish. However, both the divergence and the curl exist for this seemingly constant field. If we look a little closer, however, it is obvious why. The direction $\hat{\rho}$ is always spreading out (diverging) from the origin. The direction $\hat{\phi}$ is always twisting (curling) around the z-axis. Though the magnitude is constant they are both diverging or curling respectively

b) $\nabla \times \vec{A} = \frac{c}{r\tan\theta}\hat{r} - \frac{c}{r}\hat{\theta} + \frac{b}{r}\hat{\phi}$

$\nabla \cdot \vec{A} = \frac{2a}{r} + \frac{b}{r\tan\theta}$

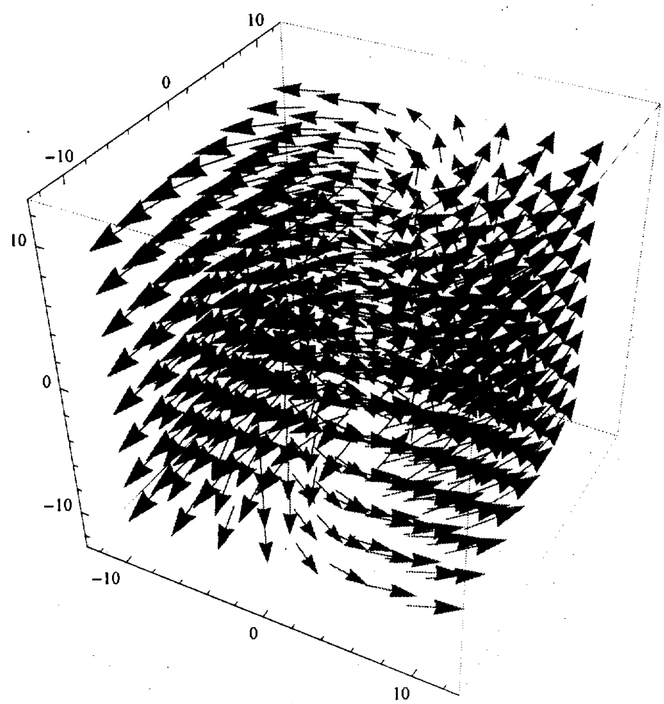
Same method and similar explanation as above

Graph 2 a)

```
In[11]:= cosphi := x / (x^2 + y^2) ^ .5
sinphi := y / (x^2 + y^2) ^ .5
costheta := z / (x^2 + y^2 + z^2) ^ .5
sintheta := (x^2 + y^2) ^ .5 / (x^2 + y^2 + z^2) ^ .5
a := 1
b := 2
c := .01
VectorPlot3D[{a * cosphi - a * sinphi, a * sinphi + b * cosphi, c * z},
 {x, -10, 10}, {y, -10, 10}, {z, -10, 10}]
```

all unit vectors must be in terms of cartesian comps. Mathematica lives in a linear world

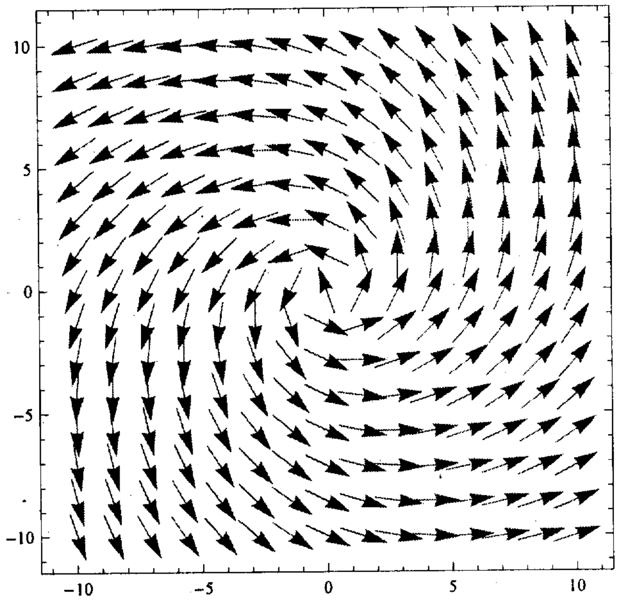
Out[18]=



PS 5

2 a) no z comp

```
VectorPlot[{a * cosphi - b * sinphi, a * sinphi + b * cosphi}, {x, -10, 10}, {y, -10, 10}]
```

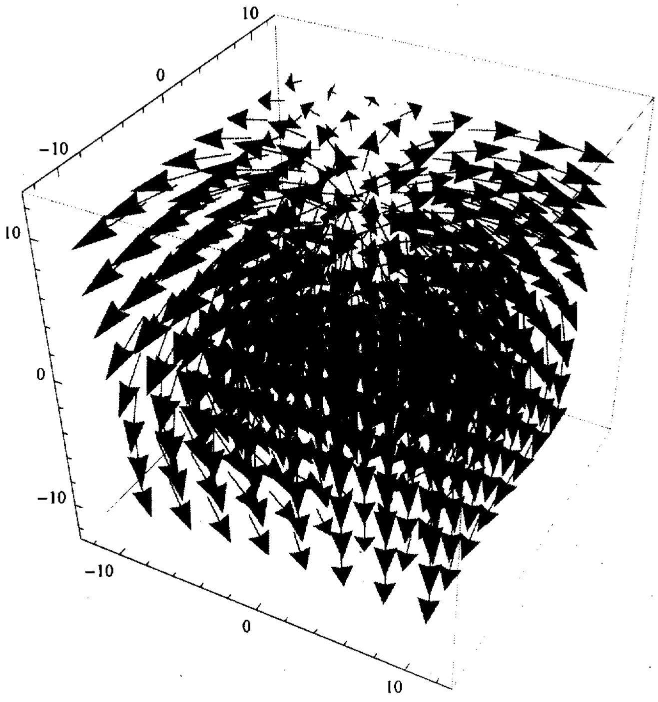


2 b)

PS 6

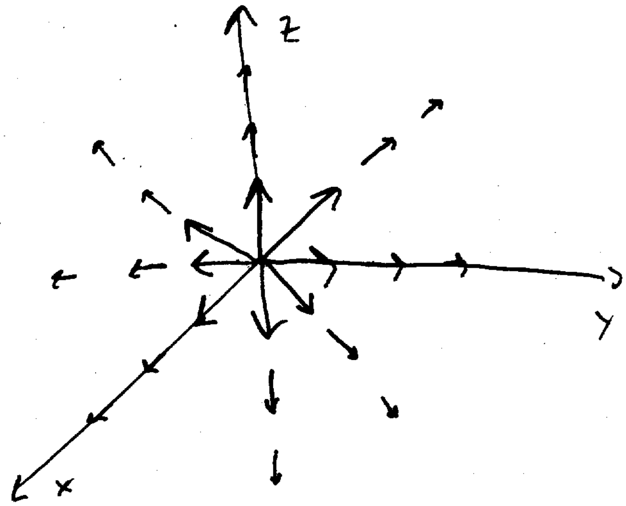

```
In[19]:= VectorPlot3D[{a * sintheta * cosphi + b * costheta * cosphi - c * sinphi,
   a * sintheta * sinphi + b * costheta * sinphi + c * cosphi, a * costheta - b * sintheta},
  {x, -10, 10}, {y, -10, 10}, {z, -10, 10}]
```

Out[19]=



3. Griffiths 1.16

$$\vec{V} = \frac{\hat{r}}{r^2}$$



Looks like a pt charge

$$\nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right)$$ since jus $\hat{r}$ comp

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{1}{\cancel{r^2}}\right) = 0$$

The field is obviously diverging but the math says no. All the stuff that made the field was at a singularity (a non-differentiable pt.) The field <u>does</u> diverge, the math just chokes. The δ function saves the day. See Griffiths section 1.5.1

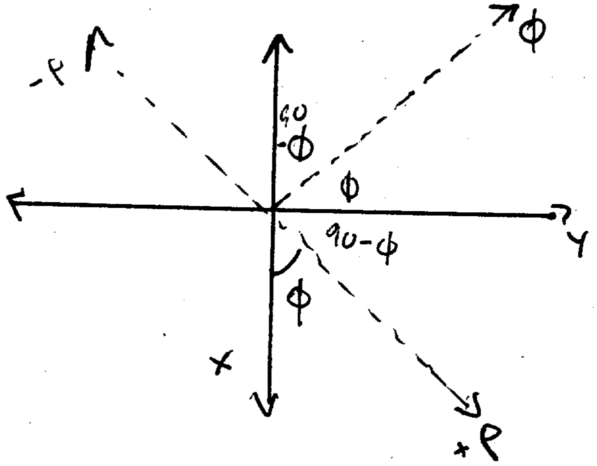
4.

cylindrical case

$z = \hat{z}$ easy

$\hat{x} = \cos\phi\, \hat{\rho} - \sin\phi\, \hat{\phi}$

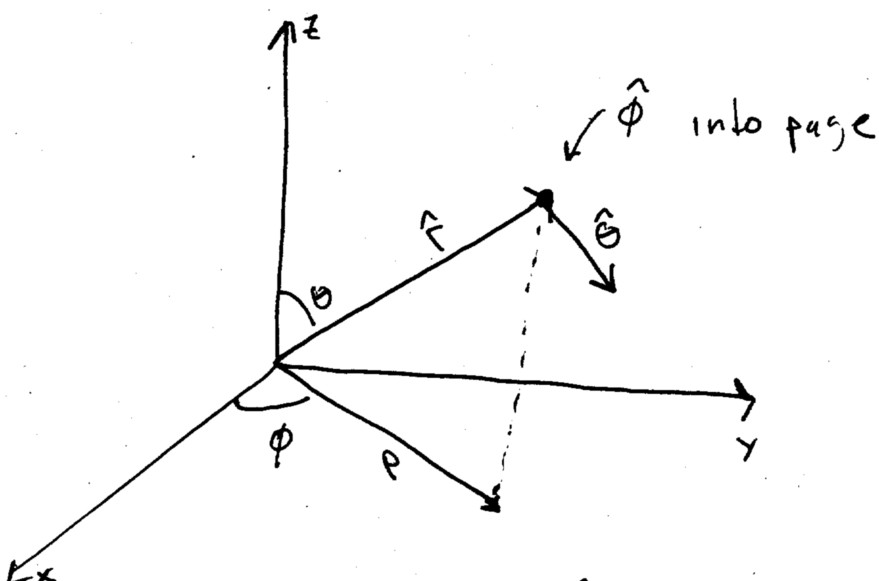
$\hat{y} = \sin\phi\, \hat{\rho} + \cos\phi\, \hat{\phi}$



from geometry just like we did in class, but backwards

spherical

Fig 1



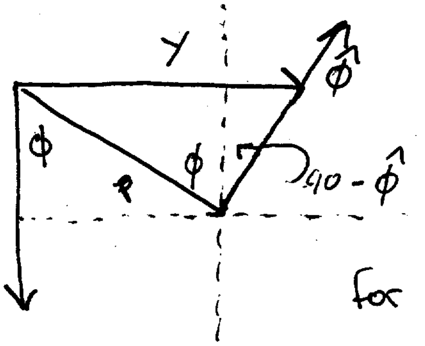
$\hat{x} = \cos\phi \sin\theta\, \hat{r} + -\sin\hat{\phi}$

gets to ρ

gets up along $\hat{r}$

$+ \underbrace{\qquad}\ \hat{\theta}$

Fig 2

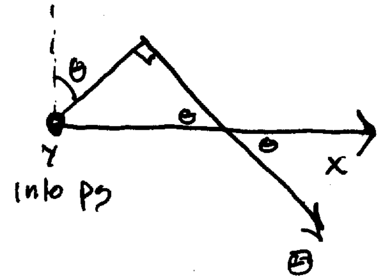


for $\hat{\phi}$ term

$\hat{\theta}$ comp $\cos\phi \cos\theta$

along $\hat{\theta}$

$\hat{x}$ axis along ρ



So $\hat{x} = \cos\phi \sin\theta\, \hat{r} + \cos\phi \cos\theta\, \hat{\theta} - \sin\phi\, \hat{\phi}$

$\hat{y} = \sin\theta \sin\phi\, \hat{r} + \cos\theta \sin\phi\, \hat{\theta} + \cos\phi\, \hat{\phi}$

gets along r just like previous

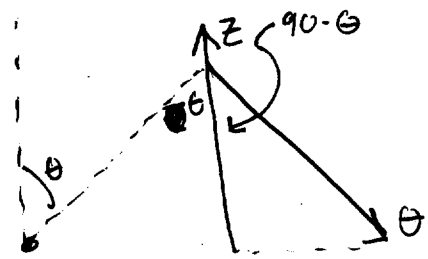
see fig 2

gets top

notice that $\hat{r}$ and $\hat{\theta}$ are same for $\hat{x}, \hat{y}$
just that $\cos\phi \to \sin\phi$

$\hat{z} = \cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}$

gets $\hat{z}$ onto $\hat{r}$



5. $du = d\vec{s} \cdot \nabla U$

$d\vec{s}$ cylindrical $= d\rho\,\hat{\rho} + \rho\,d\phi\,\hat{\phi} + dz\,\hat{z}$

$$dU_{cylin} = \frac{\partial U}{\partial \rho} d\rho + \frac{\partial U}{\partial \phi} d\phi + \frac{\partial U}{\partial z} dz$$

$$\nabla U = \frac{\partial U}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\hat{\phi} + \frac{\partial U}{\partial z}$$

From inspection

<u>For sphere</u>

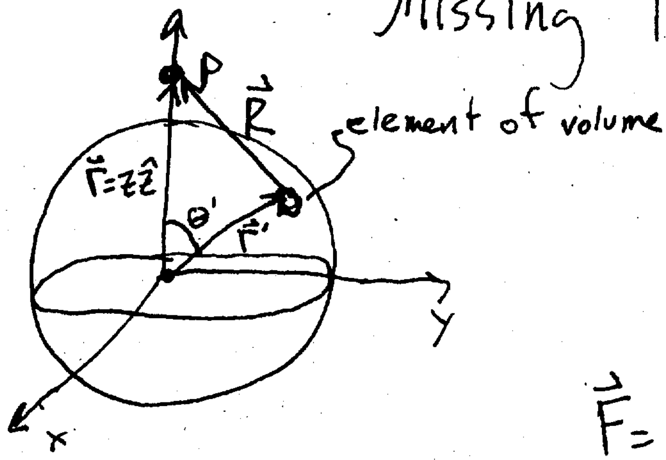
$$d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$du = \frac{\partial U}{\partial r} dr + \frac{\partial U}{\partial \theta} d\theta + \frac{\partial U}{\partial \phi} d\phi$$

$$\nabla U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

Missing Page

6.



Note $\vec{R}$ (big $\vec{R}$) is the same as Griffith's $\vec{\mathscr{r}}$ (script $\vec{\mathscr{r}}$)

$$\vec{F} = Q\vec{E}$$

From Coulombs Law and Superposition

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{R}\,\rho(r')\,d\tau'}{R^2} \qquad \text{eq. 2.8 Griffiths}$$

since $\hat{R} = \frac{\vec{R}}{R}$, we can re-write this as

$$E = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r)\,\vec{R}\,d\tau'}{R^3}.$$

We just need to find $\vec{R}$ and its length R.

We have spherical symmetry, so we integrate in spherical coords. $d\tau' = r'^2 \sin\theta'\, d\theta' d\phi' dr$. However we want the unit vectors in cartesian.

$$\vec{R} = \vec{r} - \vec{r}' = z\hat{z} - r'\hat{r} \longleftarrow \text{re-write in cartesian}$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\vec{R} = -r'\sin\theta\cos\phi\,\hat{x} - r'\sin\theta'\sin\phi'\,\hat{y} + (z - r'\cos\theta')\,\hat{z}$$

$$R = \sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2}$$

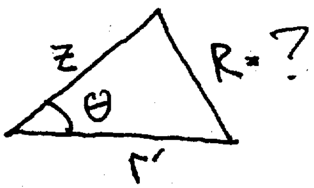
$$R = \sqrt{r'^2 \sin^2\theta \cos^2\phi + r'^2 \sin^2\theta \sin^2\phi + z^2 + r'^2\cos^2\theta - 2zr'\cos\theta'}$$

$$= \sqrt{r'^2 \sin^2\theta \underbrace{(\cos^2\phi + \sin^2\phi)}_{1} + z^2 + r'^2\cos^2\theta - 2zr'\cos\theta'}$$

$$= \sqrt{r'^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{1} + z^2 - 2zr'\cos\theta'}$$

$$= \sqrt{r'^2 + z^2 - 2r'z\cos\theta}$$

law of cosines
for any triangle



for $\theta = 90°$, pythagorian Thm.

so $F = \dfrac{Q\rho}{4\pi\varepsilon_0} \displaystyle\int_V \dfrac{\vec{R}}{R^3}\, d\tau'$ since $\hat{R} = \dfrac{\vec{R}}{R}$

$$= \frac{Q\rho}{4\pi\varepsilon_0} \int \frac{-r'\sin\theta'\cos\phi'\,\hat{x} - r'\sin\theta'\sin\phi'\,\hat{y} + (z - r'\cos\theta')\hat{z}\, d\tau'}{(z^2 + r'^2 - 2zr'\cos\theta)^{3/2}}$$

Use cartesian for vectors, but spherical for integration.

$$F_z = \frac{Q\rho}{4\pi\varepsilon_0} \int_0^a \int_0^\pi \int_0^{2\pi} \frac{(z - r'\cos\theta')}{(z^2 + r'^2 - 2zr'\cos\theta')^{3/2}}\, r'^2 \sin\theta'\, d\phi'\, d\theta'\, dr'$$

$$= \frac{Q\rho}{2\varepsilon_0} \int_0^a \int_0^\pi \frac{(z - r'\cos\theta') r'^2 \sin\theta'\, d\theta'\, dr'}{(z^2 + r'^2 - 2zr'\cos\theta')^{3/2}}$$

Let $u = \cos\theta'$, $du = -\sin\theta' d\theta'$

$$= -\frac{Q\rho}{2\varepsilon_0} \int_0^a \underbrace{\int_1^{-1} \frac{(z - r'u) r'^2\, du}{(z^2 + r'^2 - 2zr'u)^{3/2}}}\, dr'$$

integral table $\longrightarrow$ $\dfrac{zu - r'}{z^2(z^2 + r'^2 - 2zr'u)^{1/2}}\Bigg|_1^{-1} = -\dfrac{2}{z^2}$

(6 cont.)

so $F_z = \frac{Q\rho}{2\varepsilon_0} \int_0^a \frac{2 r'^2}{z^2} \, dr'$

$$\vec{F} = \boxed{\frac{Q \rho a^3}{3 \varepsilon_0 z^2} \hat{z}}$$

Note $F_x = F_y = 0$
just do ϕ' integral first, wipes them out

$Q_{Tot} = \int \rho \, d\tau'$ $\quad \rho = \text{const} \quad Q_{Tot} = \rho \int_0^a \int_0^\pi \int_0^{2\pi} r'^2 \, d\phi' \sin\theta' d\theta' dr'$

$$Q_{Tot} = \rho \frac{4}{3} \pi a^3$$

$$\text{so } \rho = \frac{Q_{Tot}}{\frac{4}{3} \pi a^3}$$

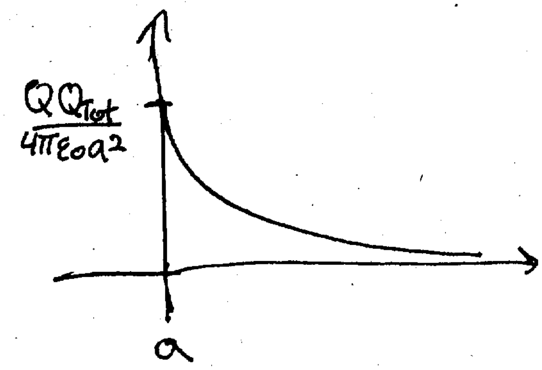
Plugging into expression above,

$$\vec{F} = \frac{Q Q_{Tot} \hat{z}}{4\pi \varepsilon_0 z^2} \quad z \to r \quad \vec{F} = \frac{Q Q_{Tot} \hat{r}}{4\pi \varepsilon_0 r^2}$$

For $z > a$ the charged sphere looks like a single charge whose value is Q_{Tot}. Just coulombs law for a big "point" charge

(6 cont.)

sketch

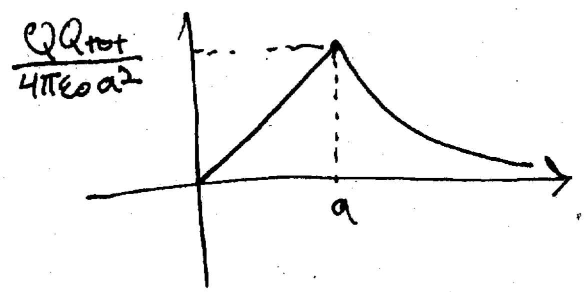


we just need to take the last integral on page 14 and change the limits. Since we are inside the sphere the r' limits go 0 to arbitrary z

$$\Rightarrow \quad F_z = \frac{Q\rho}{2\varepsilon_0}\int_0^z \frac{2r'^2}{z^2}\,dr' = \frac{Q\rho z}{3\varepsilon_0}$$

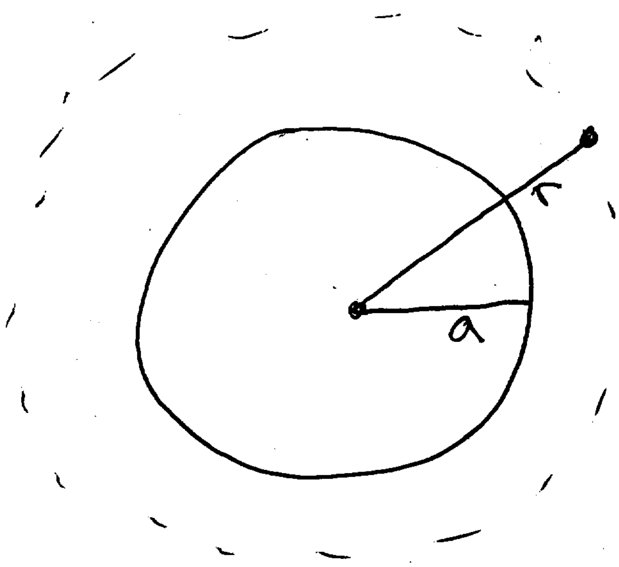
substituting $Q_{Tot} = \frac{4}{3}\pi a^3 \rho$

$$F_z = \frac{QQ_{Tot}z}{4\pi\varepsilon_0 a^3} \qquad z \to r \qquad F = \frac{QQ_{Tot}r}{4\pi\varepsilon_0 a^3}$$



6. cont

GAUSS'S LAW



$$\int \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\vec{E} = \frac{Q_{enc}}{\int da \, \varepsilon_0} \hat{r}$$

$$\boxed{\vec{E} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} \hat{r}}$$

inside and out

outside $(r > a)$ $\quad Q_{enc} = Q_{tot}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{tot}}{r^2} \hat{r}$$

inside $(r < a)$ $\quad \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{tot} \, r}{a^3} \hat{r}$

$$\rho = \frac{Q_{tot}}{\frac{4}{3}\pi a^3}$$

since for surface inside

$$Q_{enc} = \int_0^r \int_0^\pi \int_0^{2\pi} \rho \, d\tau' = \frac{4}{3}\pi r^3 \rho$$

$$\boxed{Q_{tot} = \int_0^a \int_0^\pi \int_0^{2\pi} \rho \, d\tau' = \frac{4}{3}\pi a^3 \rho}$$