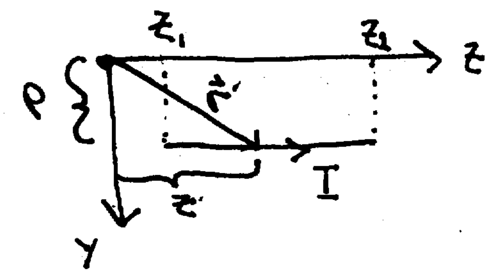
PS6 Solutions

1. Griffiths 5.22



$$\vec{r} = 0 \qquad d\vec{l} = dz\,\hat{z}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

$$= -\vec{r}\,' = -(\rho\hat{\rho} + z\hat{z})$$

$$\mathscr{r} = \sqrt{z^2 + \rho^2}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{d\vec{l}}{\mathscr{r}} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2+\rho^2}}\,\hat{z}$$

$$= \frac{\mu_0 I}{4\pi} \ln\left(z + \sqrt{z^2+\rho^2}\right)\Big|_{z_1}^{z_2}$$

$$= \frac{\mu_0 I}{4\pi} \ln\left(\frac{z_2 + \sqrt{z_2^2+\rho^2}}{z_1 + \sqrt{z_1^2+\rho^2}}\right)$$

In Figure 5.19 Griffiths $\sin\theta_1 = \dfrac{z_1}{\sqrt{z_1^2+\rho^2}}$, $\sin\theta_2 = \dfrac{z_2}{\sqrt{z_2^2+\rho^2}}$

we will use these relations

$$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A}{\partial \rho}\,\hat{\phi} \qquad \text{only component non-zero}$$

$$= -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{z_2^2+\rho^2}}\,\frac{\rho}{\sqrt{z_2^2+\rho^2}} - \frac{1}{z_1+\sqrt{z_1^2+\rho^2}}\,\frac{\rho}{\sqrt{z_1^2+\rho^2}}\right]\hat{\phi}$$

1. cont.

$$-\frac{\partial A}{\partial \rho}\hat{\phi} = -\frac{\mu_0 I \rho}{4\pi}\left[\frac{z_2 - \sqrt{z_2^2+\rho^2}}{z_2^2 - [z_2^2+\rho^2]}\frac{1}{\sqrt{z_2^2+\rho^2}} - \frac{z_1 - \sqrt{z_1^2+\rho^2}}{z_1^2 - [z_1^2+\rho^2]}\frac{1}{\sqrt{z_1^2+\rho^2}}\right]\hat{\phi}$$

$$= +\frac{\mu_0 I}{4\pi\rho}\left[\frac{z_2}{\sqrt{z_2^2+\rho^2}} - 1 - \frac{z_1}{\sqrt{z_1^2+\rho^2}} + 1\right]\hat{\phi}$$

$$= \frac{\mu_0 I}{4\pi\rho}\left[\sin\theta_2 - \sin\theta_1\right]\hat{\phi}$$

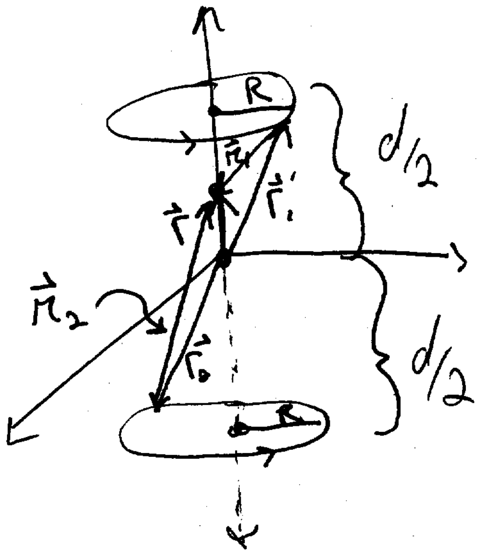
Moral of the story is that vector potential $\vec{A}$ is rarely helpful

→ is helpful in radiation

→ is helpful motivating magnetic dipole theory

Not here though, eww!

2. Griffiths 5.41



$\vec{r} = z\hat{z}$

$\vec{r}_1' = d/2\,\hat{z} + s\,\hat{s}$

$\vec{r}_2' = -\frac{d}{2}\hat{z} + s\,\hat{s}$

$\vec{I} = I\,\hat{\phi}$ $\qquad dl' = R\,d\phi'$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\vec{\mathscr{r}}_1}{\mathscr{r}_1^3}\,dl' + \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\vec{\mathscr{r}}_2}{\mathscr{r}_2^3}\,dl'$$

$\vec{\mathscr{r}}_1 = (z - d/2)\hat{z} - s\,\hat{s}$

$\vec{\mathscr{r}}_2 = (z + d/2)\hat{z} - s\,\hat{s}$ $\qquad s = R$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{\hat{\phi}\times(z - d/2)\hat{z} - R\hat{s}}{\left((z-d/2)^2 + R^2\right)^{3/2}}\,R\,d\phi' + \frac{\mu_0 I}{4\pi}\int \frac{\hat{\phi}\times(z+d/2)\hat{z} - R\hat{s}}{\left((z-d/2)^2+R^2\right)^{3/2}}\,R\,d\phi'$$

$\phi \times \hat{z} = \hat{s}$

$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$ $\qquad \int d\phi' \Rightarrow 0$

so cross products in numerator $\Rightarrow R\hat{z}$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{R\hat{z}}{\left((z-d/2)^2+R^2\right)^{3/2}}\,R\,d\phi' + \frac{\mu_0 I}{4\pi}\int \frac{R\hat{z}}{\left((z+d/2)^2+R^2\right)^{3/2}}\,R\,d\phi$$

$$\vec{B} = \frac{\mu_0 I}{2\pi}\left[\frac{R^2}{((z-d/2)^2+R^2)^{3/2}} + \frac{R^2}{((z+d/2)^2+R^2)^{3/2}}\right]\hat{z}$$

$$\frac{\partial B}{\partial z} = \frac{\mu_0 I R^2}{2\pi}\left\{\frac{-3/2\,(z-d/2)(-1)}{[(z-d/2)^2+R^2]^{5/2}} + \frac{-3/2\,(z+d/2)}{[(z+d/2)^2+R^2]^{5/2}}\right\}$$

$$\left.\frac{\partial B}{\partial z}\right|_{z=0} = \frac{3\mu_0 I R^2}{2}\left\{\frac{-d/2}{[R^2+(d/2)^2]^{5/2}} + \frac{d/2}{[R^2+(d/2)^2]^{5/2}}\right\} = 0$$

$$\frac{\partial^2 B}{\partial z^2} = \frac{3\mu_0 I R^2}{2}\left\{\frac{-1}{[R^2+(z+d/2)^2]^{5/2}} + \frac{-(z+d/2)(-5/2)(z+d/2)}{[R^2+(z+d/2)^2]^{7/2}}\right.$$

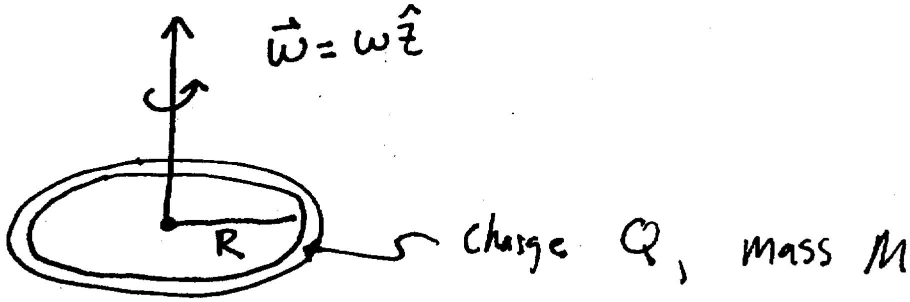
$$\left. + \frac{-1}{[R^2+(z-d/2)^2]^{5/2}} + \frac{(z-d/2)(-5/2)2(z-d/2)(-1)}{[R^2+(d/2-z)^2]^{7/2}}\right\}$$

goes to 0 when $d = R$

$$B(0) \text{ for } d=R = \frac{\mu_0 I R^2}{2}\left\{\frac{1}{(R^2+(R/2)^2)^{3/2}} + \frac{1}{(R^2+(R/2)^2)^{3/2}}\right\}$$

$$= \frac{8\mu_0 I}{5^{3/2} R}$$

3. Griffiths 5.56



a) gyromagnetic ratio $\frac{m}{L}$

$\vec{m} = I \pi R^2 \hat{z}$ $\quad$ $I = \lambda v$ $\quad$ $\lambda = \frac{Q}{2\pi R}$ $\quad$ $v = \omega R$

charge / unit length

$$I = \frac{Q}{2\pi R} \cdot \omega R = \frac{Q\omega}{2\pi}$$

$\vec{L} = I_{\text{moment}} \vec{\omega} = M R^2 \omega \hat{z}$

moment of inertia for thin ring or disk

$$\frac{m}{L} = \frac{Q\omega \pi R^2}{2\pi M R^2 \omega} = \frac{Q}{2M}$$

c) $m = \text{gyromag} \cdot L$ $\quad$ for electron ~~[illegible]~~

$L = \frac{1}{2}\hbar$, $\text{gyromag} = \frac{e}{2m_e}$ $\quad$ $m = \frac{e\hbar}{4m_e}$

c) cont.

$$\frac{e\hbar}{4m} = \frac{(1.6\times10^{-19}\,C)(1.05\times10^{-34}\,J\cdot s)}{4\,(9.11\times10^{-31}\,Kg)}$$

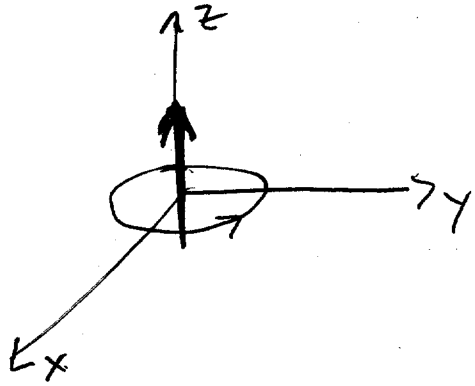
$J = N\cdot m$

Units $\dfrac{C\cdot \cancel{Kg}\cdot m^2\cdot \cancel{s}}{s^{\cancel{2}}\,\cancel{Kg}} = \dfrac{C}{s}\cdot m^2$ $\quad N = \dfrac{kg\cdot m}{s^2}$

$= A\cdot m^2$

$= 4.61\times10^{-24}\ A\,m^2$

4. Griffiths 5.34



a) $\vec{m} = I\vec{a}$

$= I\pi R^2 \hat{z}$

b) $$\vec{B} = \frac{\mu_0 I R^2}{4 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\vec{A} = \frac{\mu_0 I R^2}{4 r^2}\sin\theta\,\hat{\phi}$$

c) when $r \to z$ $\theta = 0$

so $$\vec{B} = \frac{\mu_0 I R^2}{4 z^3}\,2\,\hat{z}$$

$$= \frac{\mu_0 I R^2}{2 z^3}\,\hat{z}$$

Ex 5.6 $$\vec{B} = \frac{\mu_0 I R^2}{2}\frac{1}{(R^2+z^2)^{3/2}} \xrightarrow{} \frac{1}{z^3}$$ when $z \gg R$ ✓

For plots use program from HW2 just change const a and b to new parameters →

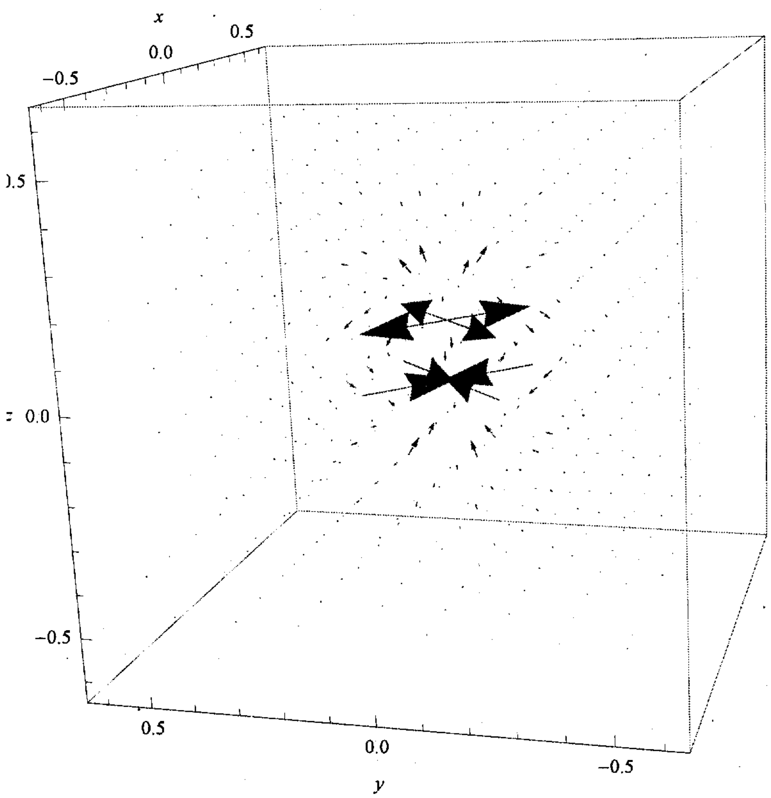
$\vec{B}$

```
In[46]:= cosphi := x / (x^2 + y^2) ^ .5
         sinphi := y / (x^2 + y^2) ^ .5
         costheta := z / (x^2 + y^2 + z^2) ^ .5
         sintheta := (x^2 + y^2) ^ .5 / (x^2 + y^2 + z^2) ^ .5
         r := (x^2 + y^2 + z^2) ^ .5
```

```
In[51]:=
         a := 2 * costheta / r^3      ← r̂ comp
         b := sintheta / r^3          ← θ̂ comp
         c := 0
```

```
In[65]:= VectorPlot3D[{a * sintheta * cosphi + b * costheta * cosphi - c * sinphi,
           a * sintheta * sinphi + b * costheta * sinphi + c * cosphi, a * costheta - b * sintheta},
          {x, -.5, .5}, {y, -.5, .5}, {z, -.5, .5}, AxesLabel → {x, y, z}]
```

Out[65]=



$\vec{A}$

```
In[67]:= a := 0
         b := 0
         c := sintheta / r^2      ← φ̂ comp
```

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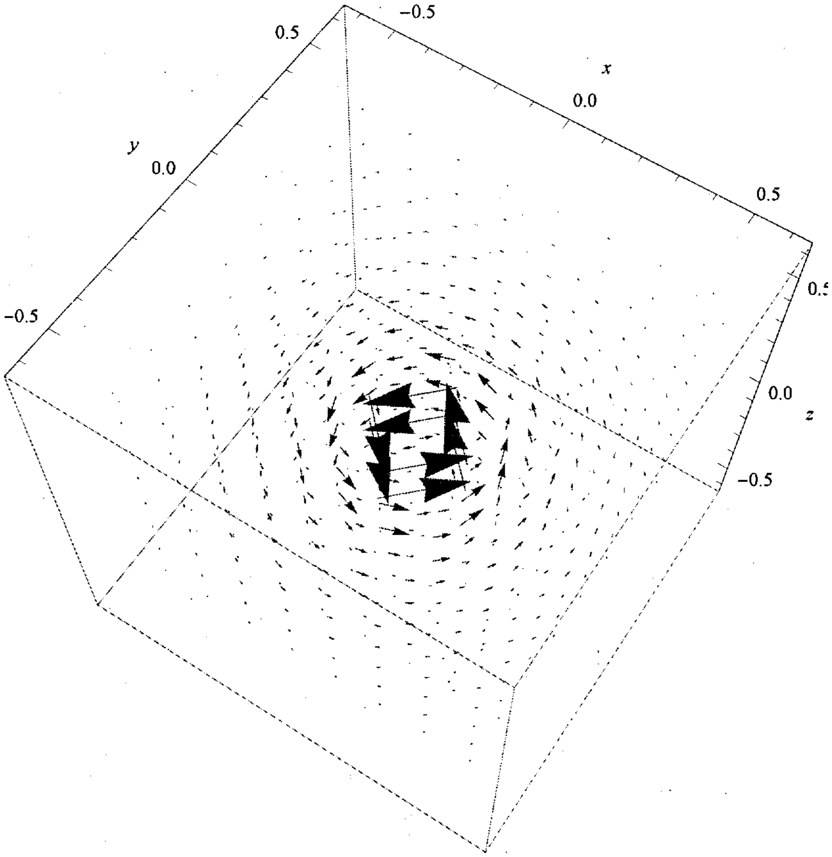
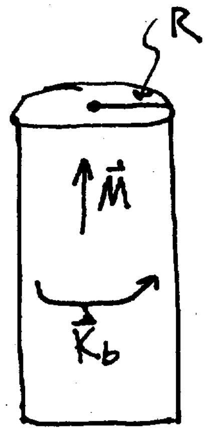
In[70]:=
```
VectorPlot3D[{a * sintheta * cosphi + b * costheta * cosphi - c * sinphi,
  a * sintheta * sinphi + b * costheta * sinphi + c * cosphi, a * costheta - b * sintheta},
 {x, -.5, .5}, {y, -.5, .5}, {z, -.5, .5}, AxesLabel → {x, y, z}]
```

Out[70]=



5 Griffiths 6.12

$\vec{M} = ks\hat{z}$



a) Hard way using normal Amperes Law $\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

$\vec{K}_b = \vec{M} \times \hat{n}$

$\hat{n}$ = normal to surface $\hat{n} = \hat{\rho}$

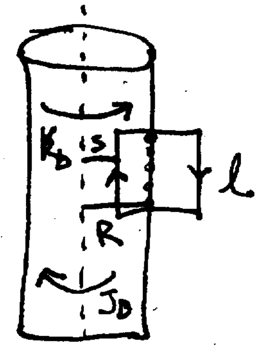
$\vec{K}_b = ks\hat{\phi}$ but only surface charge at surface

$= kR\hat{\phi}$

$J_b = \nabla \times \vec{M} = -\frac{\partial M}{\partial s}\hat{\phi} = -k\hat{\phi}$

5 a) cont.

$\vec{M} \uparrow$, $\vec{B} \uparrow$



Area inside cylinder loop encloses $= (R-s)\ell$

J_B pierces portion of loop inside the cylinder
K_b pierces loop just on edge of cylinder

$$\int \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} \qquad B\ell = \mu_0 I_{enc} \qquad \vec{B} = \frac{\mu_0 I_{enc}}{\ell}\hat{z}$$

$$I_{enc} = \int \vec{J}_b \cdot d\vec{a} + \int \vec{K}_b \cdot d\ell_\perp$$

$$= \int -k\hat{\phi}\cdot dz\,dy\,\hat{\phi} + \int kR\hat{\phi}\cdot dz\,\hat{\phi}$$

$$= -k(R-s)\ell + kR\ell = ks\ell$$

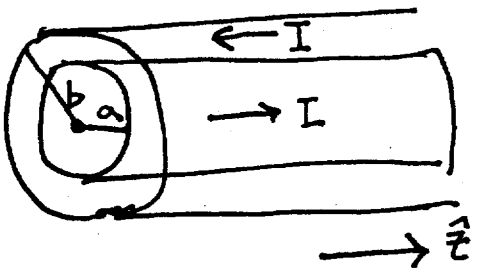
$$\vec{B} = \frac{\mu_0 k s \ell}{\ell}\hat{z} = \boxed{\mu_0 k s \hat{z}}$$

outside $\vec{B}=0$ like solenoid; essentially looks like solenoid from outside

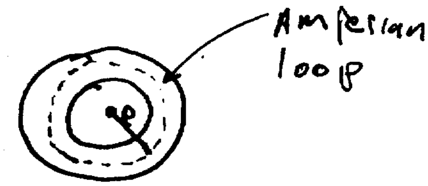
b) cheap way (right way since less work) using $\vec{H}$-field $\quad \int \vec{H}\cdot d\ell = I_{f\,enc} \quad I_{f\,enc}=0, \quad \vec{H}=0$

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ so $\boxed{\vec{B} = \mu_0\vec{M} = \mu_0 k s\hat{z}}$ ✓

6. Griffiths 6.16



coax cable



looking down coax

$$\oint \vec{H}\cdot d\vec{l} = I_{f\,enc} \qquad \vec{H} = \frac{I_f}{2\pi\rho}\hat{\phi} = \frac{I}{2\pi\rho}\hat{\phi}$$

$$\vec{B} = \mu_0(1+\chi_m)\vec{H}$$

$$\boxed{\vec{B} = \mu_0(1+\chi_m)\frac{I}{2\pi\rho}\hat{\phi}} \quad \checkmark$$

$$\vec{M} = \chi_m\vec{H} = \frac{\chi_m I}{2\pi\rho}\hat{\phi}, \qquad J_b = \nabla\times\vec{M} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho M)\hat{z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\chi_m I}{2\pi}\right)\hat{z}$$

$$= 0$$

$$K_b = \hat{M}\times\hat{n} = \frac{\chi_m I}{2\pi\rho}\hat{z}$$

sense of $\vec{\phi}$

$$= \frac{\chi_m I}{2\pi a}\hat{z} \quad \rho = a \qquad -\frac{\chi_m I}{2\pi b}\hat{z} \quad \rho = b$$

6 cont.

so $\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$ $\qquad \vec{B} = \frac{\mu_0 I_{enc}}{2\pi \rho} \hat{\phi}$

$$I_{enc} = \int \vec{J}_b \cdot d\vec{a} + \int_0^a \vec{K}_b \cdot d\vec{l}_\perp + I_f$$

$$= 0 + \frac{\chi_m I}{2\pi a} 2\pi a + I$$

$$= I(1+\chi_m)$$

$$\vec{B} = \frac{\mu_0 (1+\chi_m) I}{2\pi \rho} \hat{\phi}$$