Solutions: Sloughbeard's Treasure

1. $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = \left(A_yB_z - A_zB_y\right)\hat{\boldsymbol{x}} + \left(A_zB_x - A_xB_z\right)\hat{\boldsymbol{y}} + \left(A_xB_y - A_yB_x\right)\hat{\boldsymbol{z}}$

For all teams, $\vec{\mathbf{B}} = 6\hat{\boldsymbol{z}}$ so that $B_x = B_z = 0$, and $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = \left(A_yB_z\right)\hat{\boldsymbol{x}} - \left(A_xB_z\right)\hat{\boldsymbol{y}}$

Team 1: $\vec{\mathbf{A}} = 9.1\hat{\boldsymbol{x}} + 25.1\hat{\boldsymbol{y}} - 33.4\hat{\boldsymbol{z}}$,
Team 2: $\vec{\mathbf{A}} = -24.9\hat{\boldsymbol{x}} - 2.6\hat{\boldsymbol{y}} - 22.2\hat{\boldsymbol{z}}$
Team 3: $\vec{\mathbf{A}} = -14.9\hat{\boldsymbol{x}} + 22.1\hat{\boldsymbol{y}} + 13.7\hat{\boldsymbol{z}}$

So,

Team 1: $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = 150.6\hat{\boldsymbol{x}} - 54.6\hat{\boldsymbol{y}}$
Team 2: $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -15.6\hat{\boldsymbol{x}} + 149.4\hat{\boldsymbol{y}}$
Team 3: $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = 132.6\hat{\boldsymbol{x}} + 89.4\hat{\boldsymbol{y}}$

Note that the actual y-coordinate was opposite sign

2. $\nabla C = \frac{\partial C}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial C}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial C}{\partial z}\hat{\boldsymbol{z}}$. For all teams, $C = xy^3 + 2z^5 + 5$, so
$\nabla C = y^3\hat{\boldsymbol{x}} + 3xy^2\hat{\boldsymbol{y}} + 10z^4\hat{\boldsymbol{z}}$

For all teams, $z = 0$.

Team 1: $x = -5.21,\ y = -2.75$
Team 2: $x = 5.62,\ y = 2.96$
Team 3: $x = -0.42,\ y = 4.42$

So,

Team 1: $\nabla C = -20.8\hat{\boldsymbol{x}} - 118.2\hat{\boldsymbol{y}}$
Team 2: $\nabla C = 25.9\hat{\boldsymbol{x}} + 147.7\hat{\boldsymbol{y}}$
Team 3: $\nabla C = 86.4\hat{\boldsymbol{x}} - 24.6\hat{\boldsymbol{y}}$

3. $\nabla \times \vec{\mathbf{D}} = \left(\frac{\partial}{\partial y} D_z - \frac{\partial}{\partial z} D_y \right) \hat{\boldsymbol{x}} + \left(\frac{\partial}{\partial x} D_z - \frac{\partial}{\partial z} D_x \right) \hat{\boldsymbol{y}} + \left(\frac{\partial}{\partial x} D_y - \frac{\partial}{\partial y} D_x \right) \hat{\boldsymbol{z}}$. For all teams, $\vec{\mathbf{D}} = 3z^2 \hat{\boldsymbol{x}} + y \ln(z) \hat{\boldsymbol{y}} + yx\hat{\boldsymbol{z}}$, so $\nabla \times \vec{\mathbf{D}} = \left(x - \frac{y}{z} \right) \hat{\boldsymbol{x}} + \left(y - 6z \right) \hat{\boldsymbol{y}}$

For all teams, $z = 2$

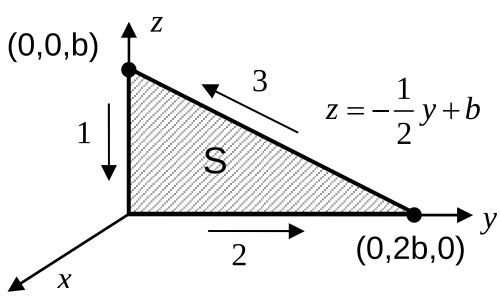
Team 1: $x = 156,\ y = 12$
Team 2: $x = 99.35,\ y = 120.1$
Team 3: $x = 98.7,\ y = 61.4$

So,

Team 1: $\nabla \times \vec{\mathbf{D}} = 150\hat{\boldsymbol{x}}$
Team 2: $\nabla \times \vec{\mathbf{D}} = 39.3\hat{\boldsymbol{x}} + 108.1\hat{\boldsymbol{y}}$
Team 3: $\nabla \times \vec{\mathbf{D}} = 68\hat{\boldsymbol{x}} + 49.4\hat{\boldsymbol{y}}$

4. Two Parts.

Part 1. *x*-comp of coordinate is equal to flux $\int_S \vec{\mathbf{E}} \bullet d\vec{\mathbf{a}}$. Note that $d\vec{\mathbf{a}} = dydz\hat{\boldsymbol{x}}$ so that all we care about is the x-comp or $\vec{\mathbf{E}}$



For all teams, $E_x = 2y$ so that

$$\int_S \vec{\mathbf{E}} \bullet d\vec{\mathbf{a}} = \int_0^{2b} \int_0^{z = -\frac{1}{2} + b} 2y dz dy$$

$$= \int_0^{2b} 2yz \Big|_0^{-\frac{1}{2} + b} dy = \int_0^{2b} 2y \left(-\frac{1}{2} y + b \right) dy$$

$$= -\frac{y^3}{3} + y^2 b \Big|_0^{2b} = \frac{4}{3} b^3$$

Part 1. cont.

Team 1: $b = 2.9$
Team 2: $b = 3.06$
Team 3: $b = 3.17$

So the x-components of the coordinates are:

Team 1: 32.5
Team 2: 38.2
Team 3: 42.5

Part 2. *y*-coord is equal to the line integral $\oint \vec{\mathbf{E}} \bullet d\vec{\ell}$. We split the path into three parts.

1. $d\vec{\ell} = dz\hat{\mathbf{z}}$
2. $d\vec{\ell} = dy\hat{\mathbf{y}}$
3. $d\vec{\ell} = dz\hat{\mathbf{z}} + dy\hat{\mathbf{y}}$, but since $z = -\frac{1}{2}y + b$, $dz = -\frac{1}{2}dy$, so that

$$d\vec{\ell} = -\frac{1}{2}dy\hat{\mathbf{z}} + dy\hat{\mathbf{y}}$$

For each team $\vec{\mathbf{E}} = 2y\hat{\mathbf{x}} + 3y^2\hat{\mathbf{y}} + Ay\hat{\mathbf{z}}$ where *A* is a different quantity for each team.

1. $\oint \vec{\mathbf{E}} \bullet d\vec{\ell} = \int_b^0 Aydz$, but on this path, $y = 0$, so $\oint \vec{\mathbf{E}} \bullet d\vec{\ell} = 0$
2. $\oint \vec{\mathbf{E}} \bullet d\vec{\ell} = \int_0^{2b} 3y^2 dy$
3. $\oint \vec{\mathbf{E}} \bullet d\vec{\ell} = \int_{2b}^0 3y^2 dy - \frac{1}{2}\int_{2b}^0 Aydy$

The integral in part 2. cancels the integral in the first part of 3. so that we are just left with

$$-\frac{1}{2}\int_{2b}^0 Aydy = -\frac{Ay^2}{4}\bigg|_{2b}^0 = Ab^2$$

For Team 1: $b = 2.9$, $A = 10.6$ so that the y-coordinate is: 89.1
For Team 2: $b = 3.06$, $A = 4.1$ so that the y-coordinate is: 38.4
For Team 3: $b = 3.17$, $A = -2.2$ so that the y-coordinate is: -22.1

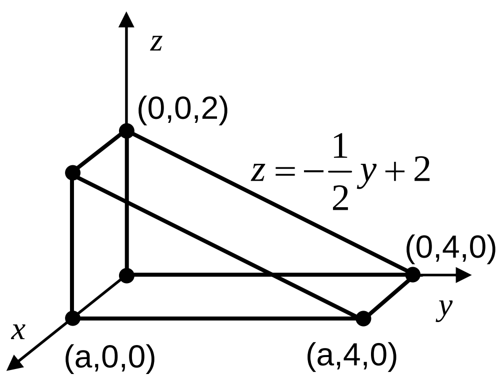
5. Two Part.

Part 1. *x*-comp of coordinate is equal of the curl $\int_S \left(\nabla\times\vec{\mathbf{E}}\right)\bullet d\vec{\mathbf{a}}$ of clue 4. multiplied by a constant.

Hopefully, you were keeping sharp and used Stokes theorem to find that $\int_S \left(\nabla\times\vec{\mathbf{E}}\right)\bullet d\vec{\mathbf{a}} = \oint \vec{\mathbf{E}}\bullet d\vec{\ell}$ for this triangle. This is of course what we used to find the y-coordinate in 4, part 2. So all you had to do here was multiply the y-coordinate from 4, part 2. times the constant given. The *x*-component of this coordinate is then

For Team 1: $89.1 * -0.56 = -49.9$
For Team 2: $38.4 * 2.25 = 86.4$
For Team 3: $-22.1 * 0.45 = -9.9$

Part 2. The y-coordinate is equal to the integral of the divergence of $\vec{\mathbf{F}}$ over the volume of the pentahedron below, $\int_V \left(\nabla\bullet\vec{\mathbf{F}}\right)d\tau$. For all teams, $\vec{\mathbf{F}} = x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + yz\hat{\mathbf{z}}$ so that $\nabla\bullet\vec{\mathbf{F}} = 3 + y$.



$$\int_V \left(\nabla\bullet\vec{\mathbf{F}}\right)d\tau = \int_0^a\int_0^4\int_0^{z=-\frac{1}{2}y+2}(3+y)\,dzdydx$$

$$= \int_0^a\int_0^4 (3+y)\left(-\frac{1}{2}y+2\right)dydx = \int_0^a\int_0^4\left(-\frac{y^2}{2}+\frac{y}{2}+6\right)dydx$$

$$= \int_0^a -\frac{y^3}{6}+\frac{y^2}{4}+6y\Big|_0^4 dx = \int_0^a \frac{52}{3}dx = \frac{52}{3}a \text{ or } \approx 17.33a$$

For Team 1: $a = 2.41$, so that the y-coord is 41.8.
For Team 2: $a = -2.43$, so that the y-coord is -42.1.
For Team 3: $a = -5.51$, so that the y-coord is -95.5.